Leray Numbers of Projections and a Topological Helly Type Theorem

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Abstract

Let X be a simplicial complex on the vertex set V. The *rational Leray number* $L(X)$ of X is the minimal d such that $\tilde{\mathrm{H}}_i(Y;\mathbb{Q})=0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$.
Suppose $V=\bigcup_{i=1}^m V_i$ is a partition of V such that the induced subcomplexes $X[V_i]$ are all 0-dimensional. Let π denote the projection of X into the $(m-1)$-simplex on the vertex set $\{1,\ldots,m\}$ given by $\pi(v)=i$ if $v \in V_i$. Let $r=\max\{|\pi^{-1}(\pi(x))| : x \in |X|\}$. It is shown that

$$L(\pi(X)) \leq rL(X)+r-1 \quad .$$

One consequence is a topological extension of a Helly type result of Amenta. Let $\mathcal{F}$ be a family of compact sets in $\mathbb{R}^d$ such that for any $\mathcal{F}' \subset \mathcal{F}$, the intersection $\bigcap \mathcal{F}'$ is either empty or contractible.
It is shown that if $\mathcal{G}$ is a family of sets such that for any finite $\mathcal{G}' \subset \mathcal{G}$, the intersection $\bigcap \mathcal{G}'$ is a union of at most r disjoint sets in $\mathcal{F}$, then the Helly number of $\mathcal{G}$ is at most $r(d+1)$.

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1 Introduction

Let $\mathcal{F}$ be a family of sets. The *Helly number* $\mathrm{h}(\mathcal{F})$ of $\mathcal{F}$ is the minimal positive integer h such that if a finite subfamily $\mathcal{K} \subset \mathcal{F}$ satisfies $\bigcap \mathcal{K}' \neq \emptyset$ for all $\mathcal{K}' \subset \mathcal{K}$ of cardinality $\leq h$, then $\bigcap \mathcal{K} \neq \emptyset$. Helly's classical theorem (1913, see e.g. [3]) asserts that the Helly number of the family of convex sets in $\mathbb{R}^d$ is $d+1$.

Helly's theorem and its numerous extensions are of central importance in discrete and computational geometry (see [3, 10]). It is of considerable interest to understand the role of convexity in these results, and to find suitable topological extensions. Indeed, it is often the case that topological methods provide a deeper understanding of the underlying combinatorics behind Helly type theorems. Helly himself realized in 1930 (see [3]) that in his theorem, convex sets can be replaced by topological cells if you impose the additional requirement that all non-empty intersections of these cells are again topological cells. Helly's topological version of his theorem also follows from the later nerve theorems of Borsuk, Leray and others (see below).

The following result was conjectured by Grünbaum and Motzkin [8] , and proved by Amenta [1]. A family of sets $\mathcal{G}$ is an *$(\mathcal{F}, r)$-family* if for any finite $\mathcal{G}' \subset \mathcal{G}$, the intersection $\bigcap \mathcal{G}'$ is a union of at most r disjoint sets from $\mathcal{F}$.

Theorem 1.1 (Amenta)**.** *Let $\mathcal{F}$ be the family of compact convex sets in $\mathbb{R}^d$. Then for any $(\mathcal{F}, r)$-family $\mathcal{G}$*

$$\mathrm{h}(\mathcal{G}) \leq r(d+1) \quad .$$

The main motivation for the present paper was to find a topological extension of Amenta's Theorem.

Let X be a simplicial complex on the vertex set V. The *induced* subcomplex on a subset of vertices $S \subset V$ is $X[S] = \{\sigma \in X : \sigma \subset S\}$. The *link* of a subset $A \subset V$ is $\mathrm{lk}(X, A) = \{\tau \in X : \tau \cup A \in X, \tau \cap A = \emptyset\}$. The geometric realization of X is denoted by $|X|$. We identify X and $|X|$ when no confusion can arise. All homology groups considered below are with rational coefficients, i.e. $\mathrm{H}_i(X) = \mathrm{H}_i(X; \mathbb{Q})$ and $\tilde{\mathrm{H}}_i(X) = \tilde{\mathrm{H}}_i(X; \mathbb{Q})$.

The *rational Leray number* $L(X)$ of X is the minimal d such that $\tilde{\mathrm{H}}_i(Y) = 0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$. The Leray number can be regarded as a simple topologically based "complexity measure" of X. Note that $L(X) = 0$ iff X is a simplex, and $L(X) \leq 1$ iff X is the clique complex

of a chordal graph (see [9]). It is well-known (see e.g. [7]) that $L(X) \le d$ iff $\tilde{\mathrm{H}}_i(\mathrm{lk}(X,\sigma)) = 0$ for all $\sigma \in X$ and $i \ge d$. Leray numbers have also significance in commutative algebra, since $L(X)$ is equal to the Castelnuovo-Mumford regularity of the Stanley-Reisner ring of X over $\mathbb{Q}$ (see [7]).

From now on we assume that $V_1, \ldots, V_m$ are finite disjoint 0-dimensional complexes, and denote their join by $V_1 * \cdots * V_m$. Let Δ_{m-1} be the simplex on the vertex set $[m] = \{1, \ldots, m\}$, and let π denote the simplicial projection from $V_1 * \cdots * V_m$ onto Δ_{m-1} given by $\pi(v) = i$ if $v \in V_i$. For a subcomplex $X \subset V_1 * \cdots * V_m$, let $r(X,\pi) = \max\{|\pi^{-1}(\pi(x))| : x \in |X|\}$. Our main result is the following

Theorem 1.2. *Let $Y = \pi(X)$ and $r = r(X,\pi)$. Then*

$$L(Y) \le rL(X) + r - 1 \quad . \tag{1}$$

Example: For $r \ge 1, d \ge 2$ let $m = rd$, and consider a partition $[m] = \bigcup_{k=1}^{r} A_k$ with $|A_k| = d$. For $i \in [m]$ let $V_i = \{i\} \times [r]$. Denote by $\Delta(A)$ the simplex on vertex set A, with boundary $\partial\Delta(A) \simeq S^{|A|-2}$. For $k, j \in [r]$ let $A_{kj} = A_k \times \{j\}$, and let

$$X_k = \Delta(A_{1k}) * \cdots * \Delta(A_{k-1,k}) * \partial\Delta(A_{kk}) * \Delta(A_{k+1,k}) * \cdots * \Delta(A_{rk}) \quad .$$

Let $X = \bigcup_{k=1}^{r} X_k$. Then $L(X) = d - 1$, and the projection $\pi : X \to \Delta_{m-1}$ satisfies $r(X,\pi) = r$. Since $\pi(X) = \partial\Delta_{m-1}$, it follows that $L(\pi(X)) = m - 1$. Hence equality is attained in (1).

As mentioned earlier, Theorem 1.2 is motivated by an application in combinatorial geometry. The *nerve* $N(\mathcal{F})$ of a family of sets $\mathcal{F}$, is the simplicial complex whose vertex set is $\mathcal{F}$ and whose simplices are all $\mathcal{F}' \subset \mathcal{F}$ such that $\bigcap \mathcal{F}' \neq \emptyset$. It is easy to see that

$$\mathrm{h}(\mathcal{F}) \le 1 + L(N(\mathcal{F})). \tag{2}$$

A finite family $\mathcal{F}$ of compact sets in some topological space is a *good cover* if for any $\mathcal{F}' \subset \mathcal{F}$, the intersection $\bigcap \mathcal{F}'$ is either empty or contractible. If $\mathcal{F}$ is a good cover in $\mathbb{R}^d$, then by the Nerve Lemma (see e.g. [2]) $L(N(\mathcal{F})) \le d$, hence follows the Topological Helly's Theorem: $\mathrm{h}(\mathcal{F}) \le d + 1$. Theorem 1.2 implies a similar topological generalization of Amenta's theorem.

Theorem 1.3. *Let $\mathcal{F}$ is a good cover in $\mathbb{R}^d$. Then for any $(\mathcal{F}, r)$-family $\mathcal{G}$*

$$\mathrm{h}(\mathcal{G}) \le r(d+1) \quad .$$

The proof of Theorem 1.2 combines a vanishing theorem for the multiple point sets of a projection, with an application of the image computing spectral sequence due to Goryunov and Mond [5]. In Section 2 we describe the Goryunov-Mond result. In Section 3 we prove our main result, Proposition 3.1, which is then used to deduce Theorem 1.2. The proof of Theorem 1.3 is given in Section 4.

2 The Image Computing Spectral Sequence

For $X \subset V_1 * \cdots * V_m$ and $k \geq 1$ define the *multiple point set* M_k by

$$M_k = \{(x_1, \ldots, x_k) \in |X|^k \ : \ \pi(x_1) = \cdots = \pi(x_k)\} \ .$$

Let W be a $\mathbb{Q}$-vector space with an action of the symmetric group S_k. Denote $\mathrm{Alt} = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\sigma \in \mathbb{Q}[S_k]$. Then

$$\begin{aligned} \mathrm{Alt}\, W = \{\mathrm{Alt}\, w : w \in W\} = \\ \{w \in W : \sigma w = \mathrm{sign}(\sigma) w \ for\ all\ \sigma \in S_k\} \ . \end{aligned} \tag{3}$$

The natural action of S_k on M_k induces an action on the rational chain complex $\mathrm{C}_*(M_k)$ and on the rational homology $\mathrm{H}_*(M_k)$. The idempotence of Alt implies that

$$\mathrm{Alt}\, \mathrm{H}_*(M_k) \cong \mathrm{H}_*(\mathrm{Alt}\, \mathrm{C}(M_k)) \ . \tag{4}$$

The following result is due to Goryunov and Mond [5] (see also [4] and [6]).

Theorem 2.1 (Goryunov and Mond)**.** *Let $Y = \pi(X)$ and $r = r(X, \pi)$. Then there exists a homology spectral sequence $\{E^r_{p,q}\}$ converging to $\mathrm{H}_*(Y)$ with*

$$E^1_{p,q} = \begin{cases} \mathrm{Alt}\, \mathrm{H}_q(M_{p+1}) & 0 \leq p \leq r-1, 0 \leq q \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Remark: The E^1 terms in the original formulation of Theorem 2.1 in [5], are given by $E^1_{p,q} = \mathrm{Alt}\, \mathrm{H}_q(D^{p+1})$ where

$$D^k = \mathrm{closure}\{(x_1, \ldots, x_k) \in |X|^k : \pi(x_1) = \cdots = \pi(x_k), x_i \neq x_j \text{ for } i \neq j\} \ .$$

The isomorphism

$$\mathrm{Alt}\, \mathrm{H}_q(D^{p+1}) \cong \mathrm{Alt}\, \mathrm{H}_q(M_{p+1})$$

which implies (5), is proved in Theorem 3.4 in [6]. Indeed, as noted there, the inclusion $D^{p+1} \to M_{p+1}$ induces an isomorphism $\mathrm{Alt}\, \mathrm{C}_q(D^{p+1}) \cong \mathrm{Alt}\, \mathrm{C}_q(M_{p+1})$ already at the alternating chains level.

3 Homology of the Multiple Point Set

In this section we study the homology of a generalization of the multiple point set. For subcomplexes $X_1, \ldots, X_k \subset V_1 * \cdots * V_m$, let

$$M(X_1, \ldots, X_k) = \{(x_1, \ldots, x_k) \in |X_1| \times \cdots \times |X_k| : \pi(x_1) = \cdots = \pi(x_k)\} .$$

In particular, if $X_1 = \cdots = X_k = X$ then $M(X_1, \ldots, X_k) = M_k$.

We identify the generalized multiple point set $M(X_1, \ldots, X_k)$ with the simplicial complex whose p-dimensional simplices are $\{w_{i_0}, \ldots, w_{i_p}\}$, where $1 \le i_0 < \cdots < i_p \le m$, $w_{i_j} = (v_{i_j,1}, \ldots, v_{i_j,k}) \in V_{i_j}^k$ and $\{v_{i_0,r}, \ldots, v_{i_p,r}\} \in X_r$ for all $1 \le r \le k$. The main ingredient in the proof of Theorem 1.2 is the following

Proposition 3.1. *$\tilde{\mathrm{H}}_j\big(M(X_1, \ldots, X_k)\big) = 0$ for $j \ge \sum_{i=1}^k L(X_i)$.*

The proof of Proposition 3.1 depends on a spectral sequence argument given below. We first recall some definitions. Let K be a simplicial complex. The subdivision $\mathrm{sd}(K)$ is the order complex of the set of the non-empty simplices of K ordered by inclusion. For $\sigma \in K$ let $D_K(\sigma)$ denote the order complex of the interval $[\sigma, \cdot] = \{\tau \in K \ : \ \tau \supset \sigma\}$. Let $\dot{D}_K(\sigma)$ denote the order complex of the interval $(\sigma, \cdot] = \{\tau \in K \ : \ \tau \supsetneqq \sigma\}$. Note that $\dot{D}_K(\sigma)$ is isomorphic to $\mathrm{sd}(\mathrm{lk}(K, \sigma))$ via the simplicial map $\tau \to \tau - \sigma$. Since $D_K(\sigma)$ is contractible, it follows that $\mathrm{H}_i\big(D_K(\sigma), \dot{D}_K(\sigma)\big) \cong \tilde{\mathrm{H}}_{i-1}\big(\mathrm{lk}(K, \sigma)\big)$ for all $i \ge 0$.

For $\sigma \in V_1 * \cdots * V_m$, let $\tilde{\sigma} = \bigcup_{i \in \pi(\sigma)} V_i$. Note that if $\sigma_2 \in X_2, \ldots, \sigma_k \in X_k$ then there is an isomorphism

$$M(X_1, \sigma_2, \ldots, \sigma_k) \cong X_1[\cap_{i=2}^k \tilde{\sigma}_i] \quad . \tag{6}$$

For $0 \le p \le n = \sum_{i=2}^k \dim X_i$ let

$$\mathcal{S}'_p = \{(\sigma_2, \ldots, \sigma_k) \in X_2 \times \cdots \times X_k \ : \ \sum_{i=2}^k \dim \sigma_i \ge n - p\}$$

and let $\mathcal{S}_p = \mathcal{S}'_p - \mathcal{S}'_{p-1}$. For $\underline{\sigma} = (\sigma_2, \ldots, \sigma_k) \in \mathcal{S}'_p$ let

$$A_{\underline{\sigma}} = M(X_1, \sigma_2, \ldots, \sigma_k) \times D_{X_2}(\sigma_2) \times \cdots \times D_{X_k}(\sigma_k) \quad ,$$

$$B_{\underline{\sigma}} = M(X_1,\sigma_2,\ldots,\sigma_k) \times \Big(\bigcup_{j=2}^{k} D_{X_2}(\sigma_2)\times\cdots\times \dot{D}_{X_j}(\sigma_j)\times\cdots\times D_{X_k}(\sigma_k)\Big) \quad .$$

Proposition 3.2. *There exists a homology spectral sequence $\{E^r_{p,q}\}$ converging to $\mathrm{H}_*\big(M(X_1,\ldots,X_k)\big)$ such that*

$$E^1_{p,q} = \bigoplus_{\underline{\sigma}\in\mathcal{S}_p} \bigoplus_{\substack{i_1,\ldots,i_k\geq 0\\ i_1+\cdots+i_k=p+q}} \mathrm{H}_{i_1}\big(X_1[\cap_{i=2}^{k}\tilde{\sigma}_i]\big)\otimes \bigotimes_{j=2}^{k}\tilde{\mathrm{H}}_{i_j-1}\big(\mathrm{lk}(X_j,\sigma_j)\big) \tag{7}$$

for $0\leq p\leq n$, $0\leq q$, *and* $E^1_{p,q}=0$ *otherwise.*

Proof: For $0\leq p\leq n$ let

$$K_p = \bigcup_{\underline{\sigma}\in\mathcal{S}'_p} A_{\underline{\sigma}} \subset M(X_1,\ldots,X_k)\times \mathrm{sd}(X_2)\times\cdots\times\mathrm{sd}(X_k).$$

Write $K=K_n$, and consider the projection on the first coordinate $\theta: K\to M(X_1,\ldots,X_k)$. Let $(x_1,\ldots,x_k)\in M(X_1,\ldots,X_k)$, and let σ_i denote the minimal simplex in X_i that contains x_i. Then the fiber

$$\theta^{-1}\big((x_1,\ldots,x_k)\big) = \{(x_1,\ldots,x_k)\}\times D_{X_2}(\sigma_2)\times\cdots\times D_{X_k}(\sigma_k)$$

is a cone, hence K is homotopy equivalent to $M(X_1,\ldots,X_k)$. The filtration $\emptyset\subset K_0\subset\cdots\subset K_n=K$ gives rise to a homology spectral sequence $\{E^r_{p,q}\}$ converging to $\mathrm{H}_*(K)\cong\mathrm{H}_*(M(X_1,\ldots,X_m))$. The $E^1_{p,q}$ terms are computed as follows. First note that

$$\bigcup_{\underline{\sigma}\in\mathcal{S}_p} A_{\underline{\sigma}} \bigcap K_{p-1} = \bigcup_{\underline{\sigma}\in\mathcal{S}_p} B_{\underline{\sigma}} \quad . \tag{8}$$

Secondly, $\big(A_{\underline{\sigma}}-B_{\underline{\sigma}}\big)\cap A_{\underline{\sigma}'}=\emptyset$ for $\underline{\sigma}\neq\underline{\sigma}'\in\mathcal{S}_p$. Hence

$$\mathrm{H}_*\Big(\bigcup_{\underline{\sigma}\in\mathcal{S}_p} A_{\underline{\sigma}}, \bigcup_{\underline{\sigma}\in\mathcal{S}_p} B_{\underline{\sigma}}\Big) \cong \bigoplus_{\underline{\sigma}\in\mathcal{S}_p} \mathrm{H}_*(A_{\underline{\sigma}},B_{\underline{\sigma}}) \quad . \tag{9}$$

Applying excision, (8),(9), and the Künneth formula we obtain:

$$E^1_{p,q} = \mathrm{H}_{p+q}(K_p,K_{p-1}) \cong \mathrm{H}_{p+q}\Big(\bigcup_{\underline{\sigma}\in\mathcal{S}_p} A_{\underline{\sigma}}, \bigcup_{\underline{\sigma}\in\mathcal{S}_p} A_{\underline{\sigma}}\bigcap K_{p-1}\Big) \cong$$

$$\mathrm{H}_{p+q}\Big(\bigcup_{\underline{\sigma}\in\mathcal{S}_p} A_{\underline{\sigma}}, \bigcup_{\underline{\sigma}\in\mathcal{S}_p} B_{\underline{\sigma}}\Big) \cong \bigoplus_{\underline{\sigma}\in\mathcal{S}_p} \mathrm{H}_{p+q}(A_{\underline{\sigma}}, B_{\underline{\sigma}}) \cong$$

$$\bigoplus_{\underline{\sigma}\in\mathcal{S}_p} \bigoplus_{\substack{i_1,\ldots,i_k\geq 0 \\ i_1+\cdots+i_k=p+q}} \mathrm{H}_{i_1}\big(M(X_1,\sigma_2,\ldots,\sigma_k)\big) \otimes \bigotimes_{j=2}^{k} \mathrm{H}_{i_j}\big(D_{X_j}(\sigma_j), \dot{D}_{X_j}(\sigma_j)\big) \cong$$

$$\bigoplus_{\underline{\sigma}\in\mathcal{S}_p} \bigoplus_{\substack{i_1,\ldots,i_k\geq 0 \\ i_1+\cdots+i_k=p+q}} \mathrm{H}_{i_1}\big(X_1[\cap_{i=2}^k \tilde{\sigma}_i]\big) \otimes \bigotimes_{j=2}^{k} \tilde{\mathrm{H}}_{i_j-1}\big(\mathrm{lk}(X_j,\sigma_j)\big) \quad .$$

□

Proof of Proposition 3.1: If $L(X_j) = 0$ for all $1 \leq j \leq k$, then all the X_j's are simplicies, say $X_j = \sigma_j$. It follows that $M(X_1,\ldots,X_k)$ is isomorphic to the simplex $\bigcap_{j=1}^k \pi(\sigma_j)$ and thus has vanishing reduced homology in all nonnegative dimensions. Suppose then that $m = \sum_{j=1}^k L(X_j) > 0$. Without loss of generality we may assume that $L(X_1) > 0$. Let $i_1,\ldots,i_k \geq 0$ such that $\sum_{j=1}^k i_j \geq m$. Then either $i_1 \geq L(X_1)$ and then $\mathrm{H}_{i_1}\big(X_1[\cap_{i=2}^k \tilde{\sigma}_i]\big) = 0$, or there exists a $2 \leq j \leq k$ such that $i_j - 1 \geq L(X_j)$ and then $\tilde{\mathrm{H}}_{i_j-1}\big(\mathrm{lk}(X_j,\sigma_j)\big) = 0$. By (7) it follows that $E^1_{p,q} = 0$ if $p+q \geq m$, hence $\tilde{\mathrm{H}}_j\big(M(X_1,\ldots,X_k)\big) = 0$ for all $j \geq m$.

□

Remark: If all the V_j's are singletons then $M(X_1,\ldots,X_k)$ is isomorphic to $\bigcap_{j=1}^k X_j$. Hence Proposition 3.1 implies the following result of [7].

Corollary 3.3 ([7])**.** *If $X_1,\ldots,X_k$ are simplicial complexes on the same vertex set, then*

$$L(\bigcap_{j=1}^k X_j) \leq \sum_{j=1}^k L(X_j) \quad .$$

□

Proof of Theorem 1.2: Let $Y = \pi(X)$ and $r = r(X,\pi)$. Assuming as we may that $L(X) > 0$, we have to show that $\mathrm{H}_m(Y) = 0$ for $m \geq rL(X)+r-1$. By Theorem 2.1 it suffices to show that $\mathrm{Alt}\,\mathrm{H}_q(M_{p+1}) = 0$ for all pairs (p,q) such that $p \leq r-1$ and $p+q \geq rL(X)+r-1$. Indeed, $p \leq r-1$ implies that $q \geq rL(X) \geq (p+1)L(X)$, thus $\mathrm{H}_q(M_{p+1}) = 0$ by Proposition 3.1.

□

4 A Topological Amenta Theorem

Proof of Theorem 1.3: Suppose $\mathcal{G} = \{G_1, \ldots, G_m\}$ is an $(\mathcal{F}, r)$-family. Write $G_i = \bigcup_{j=1}^{r_i} F_{ij}$, where $r_i \le r$ and $F_{ij} \bigcap F_{ij'} = \emptyset$ for $1 \le j \ne j' \le r_i$. Let $V_i = \{F_{i1}, \ldots, F_{ir_i}\}$ and consider the nerve

$$X = N(\{F_{ij} : 1 \le i \le m,\ 1 \le j \le r_i\}) \subset V_1 * \cdots * V_m \quad .$$

Let Δ_{m-1} be the simplex on the vertex set $\{G_1, \ldots, G_m\}$ and let π denote the projection of $V_1 * \cdots * V_m$ into Δ_{m-1} given by $\pi(F_{ij}) = G_i$. Then $\pi(X) = N(\mathcal{G})$. Let $y \in |N(\mathcal{G})|$ and let $\sigma = \{G_i \ :\ i \in I\}$ be the minimal simplex in $N(\mathcal{G})$ such that $y \in |\sigma|$. Then

$$|\pi^{-1}(y)| = |\{\ (j_i : i \in I)\ :\ \bigcap_{i \in I} F_{ij_i} \ne \emptyset\ \}|\ . \tag{10}$$

On the other hand

$$\bigcap_{i \in I} G_i = \bigcup_{(j_i : i \in I)} \bigcap_{i \in I} F_{ij_i} \tag{11}$$

and the union on the right is a disjoint union. The assumption that $\mathcal{G}$ is an $(\mathcal{F}, r)$ family, together with (10) and (11), imply that $|\pi^{-1}(y)| \le r$ for all $y \in |N(\mathcal{G})|$. Since $\mathcal{F}$ is a good cover in $\mathbb{R}^d$, the Leray number of the nerve satisfies $L(X) = L(N(\mathcal{F})) \le d$. Therefore by (2) and Theorem 1.2

$$\mathrm{h}(\mathcal{G}) \le 1 + L(N(\mathcal{G})) = 1 + L(\pi(X)) \le$$

$$1 + rL(X) + r - 1 \le r(d+1) \quad .$$

□

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